8.5.3 Exercises

For a link to all of the additional resources available for this section, click OSttS Chapter 8 materials.

In Exercises 1 - 8, compute the determinant of the given matrix. (Some of these matrices appeared in Exercises 1 - 8 in Section 8.4.)

For help with these exercises, click one or more of the resources below:

- Finding the determinant of a 2×2 matrix
- Finding the determinant of a 3×3 matrix
- Finding the determinant of a 4×4 matrix

1. $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

2. $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

3. $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

4. $L = \begin{bmatrix} \dfrac{1}{x^3} & \dfrac{\ln(x)}{x^3} \\ -\dfrac{3}{x^4} & \dfrac{1 - 3\ln(x)}{x^4} \end{bmatrix}$

5. $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

6. $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

7. $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

8. $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

In Exercises 9 - 14, use Cramer's Rule to solve the system of linear equations.

For help with these exercises, click on the resource below:

- Solving a system of linear equations using Cramer's Rule

9. $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$

10. $\left\{ \begin{array}{rcr} 2x - 4y & = & 5 \\ 10x + 13y & = & -6 \end{array} \right.$

11. $\left\{ \begin{array}{rcr} x + y & = & 8000 \\ 0.03x + 0.05y & = & 250 \end{array} \right.$

12. $\left\{ \begin{array}{rcl} \frac{1}{2}x - \frac{1}{5}y & = & 1 \\ 6x + 7y & = & 3 \end{array} \right.$

13. $\left\{ \begin{array}{rcl} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

14. $\left\{ \begin{array}{rcr} 3x + y - 2z & = & 10 \\ 4x - y + z & = & 5 \\ x - 3y - 4z & = & -1 \end{array} \right.$

In Exercises 15 - 16, use Cramer's Rule to solve for x_4.

For help with these exercises, click on the resource below:

- Solving a system of linear equations using Cramer's Rule

15. $\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$

16. $\left\{ \begin{array}{rcr} 4x_1 + x_2 & = & 4 \\ x_2 - 3x_3 & = & 1 \\ 10x_1 + x_3 + x_4 & = & 0 \\ -x_2 + x_3 & = & -3 \end{array} \right.$

In Exercises 17 - 18, find the inverse of the given matrix using their determinants and adjoints.

For help with these exercises, click on the resource below:

- Finding the inverse of a matrix using co-factors (determinants)

17. $B = \left[\begin{array}{rr} 12 & -7 \\ -5 & 3 \end{array} \right]$

18. $F = \left[\begin{array}{rrr} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{array} \right]$

19. Carl's Sasquatch Attack! Game Card Collection is a mixture of common and rare cards. Each common card is worth \$0.25 while each rare card is worth \$0.75. If his entire 117 card collection is worth \$48.75, how many of each kind of card does he own?

20. How much of a 5 gallon 40% salt solution should be replaced with pure water to obtain 5 gallons of a 15% solution?

21. How much of a 10 liter 30% acid solution must be replaced with pure acid to obtain 10 liters of a 50% solution?

22. Daniel's Exotic Animal Rescue houses snakes, tarantulas and scorpions. When asked how many animals of each kind he boards, Daniel answered: 'We board 49 total animals, and I am responsible for each of their 272 legs and 28 tails.' How many of each animal does the Rescue board? (Recall: tarantulas have 8 legs and no tails, scorpions have 8 legs and one tail, and snakes have no legs and one tail.)

23. This exercise is a continuation of Exercise 16 in Section 8.4. Just because a system is consistent independent doesn't mean it will admit a solution that makes sense in an applied setting. Using the nutrient values given for Ippizuti Fish, Misty Mushrooms, and Sun Berries, use Cramer's Rule to determine the number of servings of Ippizuti Fish needed to meet the needs of a daily diet which requires 2500 calories, 1000 grams of protein, and 400 milligrams of Vitamin X. Now use Cramer's Rule to find the number of servings of Misty Mushrooms required. Does a solution to this diet problem exist?

24. Let $R = \begin{bmatrix} -7 & 3 \\ 11 & 2 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -5 \\ 6 & 9 \end{bmatrix}$ $T = \begin{bmatrix} 11 & 2 \\ -7 & 3 \end{bmatrix}$, and $U = \begin{bmatrix} -3 & 15 \\ 6 & 9 \end{bmatrix}$

    (a) Show that $\det(RS) = \det(R)\det(S)$

    (b) Show that $\det(T) = -\det(R)$

    (c) Show that $\det(U) = -3\det(S)$

25. For M, N, and P below, show that $\det(M) = 0$, $\det(N) = 0$ and $\det(P) = 0$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

26. Let A be an arbitrary invertible 3×3 matrix.

    (a) Show that $\det(I_3) = 1$.[8]

    (b) Using the facts that $AA^{-1} = I_3$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

The purpose of Exercises 27 - 30 is to introduce you to the eigenvalues and eigenvectors of a matrix.[9] We begin with an example using a 2×2 matrix and then guide you through some exercises using a 3×3 matrix. Consider the matrix

$$C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$$

from Exercise 2. We know that $\det(C) = 0$ which means that $CX = 0_{2\times 2}$ does not have a unique solution. So there is a nonzero matrix Y with $CY = 0_{2\times 2}$. In fact, every matrix of the form

$$Y = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix}$$

is a solution to $CX = 0_{2\times 2}$, so there are infinitely many matrices such that $CX = 0_{2\times 2}$. But consider the matrix

$$X_{41} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

It is NOT a solution to $CX = 0_{2\times 2}$, but rather,

$$CX_{41} = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 123 \\ 287 \end{bmatrix} = 41 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

[8] If you think about it for just a moment, you'll see that $\det(I_n) = 1$ for any natural number n. The formal proof of this fact requires the Principle of Mathematical Induction (Section 9.3) so we'll stick with $n = 3$ for the time being.

[9] This material is usually given its own chapter in a Linear Algebra book so clearly we're not able to tell you everything you need to know about eigenvalues and eigenvectors. They are a nice application of determinants, though, so we're going to give you enough background so that you can start playing around with them.

In fact, if Z is of the form

$$Z = \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix}$$

then

$$CZ = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{123}{7}t \\ 41t \end{bmatrix} = 41 \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = 41Z$$

for all t. The big question is "How did we know to use 41?"

We need a number λ such that $CX = \lambda X$ has nonzero solutions. We have demonstrated that $\lambda = 0$ and $\lambda = 41$ both worked. Are there others? If we look at the matrix equation more closely, what we *really* wanted was a nonzero solution to $(C - \lambda I_2)X = 0_{2\times 2}$ which we know exists if and only if the determinant of $C - \lambda I_2$ is zero.[10] So we computed

$$\det(C - \lambda I_2) = \det\left(\begin{bmatrix} 6-\lambda & 15 \\ 14 & 35-\lambda \end{bmatrix}\right) = (6-\lambda)(35-\lambda) - 14 \cdot 15 = \lambda^2 - 41\lambda$$

This is called the **characteristic polynomial** of the matrix C and it has two zeros: $\lambda = 0$ and $\lambda = 41$. That's how we knew to use 41 in our work above. The fact that $\lambda = 0$ showed up as one of the zeros of the characteristic polynomial just means that C itself had determinant zero which we already knew. Those two numbers are called the **eigenvalues** of C. The corresponding matrix solutions to $CX = \lambda X$ are called the **eigenvectors** of C and the 'vector' portion of the name will make more sense after you've studied vectors.

Now it's your turn. In the following exercises, you'll be using the matrix G from Exercise 6.

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$$

27. Show that the characteristic polynomial of G is $p(\lambda) = -\lambda(\lambda - 1)(\lambda - 22)$. That is, compute $\det(G - \lambda I_3)$.

28. Let $G_0 = G$. Find the parametric description of the solution to the system of linear equations given by $GX = 0_{3\times 3}$.

29. Let $G_1 = G - I_3$. Find the parametric description of the solution to the system of linear equations given by $G_1X = 0_{3\times 3}$. Show that any solution to $G_1X = 0_{3\times 3}$ also has the property that $GX = 1X$.

30. Let $G_{22} = G - 22I_3$. Find the parametric description of the solution to the system of linear equations given by $G_{22}X = 0_{3\times 3}$. Show that any solution to $G_{22}X = 0_{3\times 3}$ also has the property that $GX = 22X$.

[10] Think about this.

Checkpoint Quiz 8.5

1. Consider the system: $\left\{ \begin{array}{rcl} 3x + y - z & = & 9 \\ 2x - 4y - 5z & = & 4 \\ x - y - z & = & 3 \end{array} \right.$. Find y using Cramer's Rule.

For worked out solutions to this quiz, click the link below:

- Quiz Solution

8.5.4 Answers

1. $\det(B) = 1$
2. $\det(C) = 0$
3. $\det(Q) = x^2$
4. $\det(L) = \dfrac{1}{x^7}$
5. $\det(F) = -12$
6. $\det(G) = 0$
7. $\det(V) = 20i + 43j + 4k$
8. $\det(H) = -2$
9. $x = 39,\ y = -13$
10. $x = \frac{41}{66},\ y = -\frac{31}{33}$
11. $x = 7500,\ y = 500$
12. $x = \frac{76}{47},\ y = -\frac{45}{47}$
13. $x = 1,\ y = 2,\ z = 0$
14. $x = \frac{121}{60},\ y = \frac{131}{60},\ z = -\frac{53}{60}$
15. $x_4 = 4$
16. $x_4 = -1$
17. $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$
18. $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$
19. Carl owns 78 common cards and 39 rare cards.
20. 3.125 gallons.
21. $\frac{20}{7} \approx 2.85$ liters.
22. The rescue houses 15 snakes, 21 tarantulas and 13 scorpions.
23. Using Cramer's Rule, we find we need 53 servings of Ippizuti Fish to satisfy the dietary requirements. The number of servings of Misty Mushrooms required, however, is -1120. Since it's impossible to have a negative number of servings, there is no solution to the applied problem, despite there being a solution to the mathematical problem. A cautionary tale about using Cramer's Rule: just because you are guaranteed a mathematical answer for each variable doesn't mean the solution will make sense in the 'real' world.